A Derivative-Free Optimization Algorithm with Low-Dimensional Subspace Techniques for Large-Scale Problems

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Outline

1. Derivative-free optimization

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- **The problem**

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    - the function evaluation of f is expensive.

1. Derivative-free optimization

- **Importance and difficulty**

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Introduction to Derivative-Free Optimization

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- **Existing methods**
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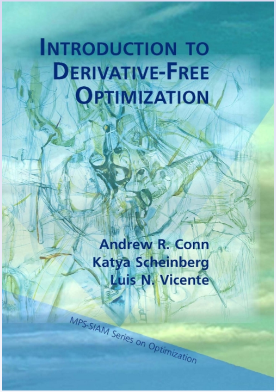
    - model-based methods, like trust-region methods.

1. Derivative-free optimization



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  - Model construction:

    $$\text{Objective Function} \xrightarrow{\text{Interplolation}} \text{Model Function.}$$

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  - What if $|\mathcal{I}| < (n+1)(n+2)/2$ (for example, $|\mathcal{I}| = O(n)$)?
  - Regularization:

$$\min_{Q \in \mathcal{Q}} \mathscr{F}(Q)$$
$$\text{s.t. } Q(x) = f(x), \quad x \in \mathcal{I}.$$

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  - difficult to exploit the structure.

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  - Solve a difficult problem by solving a sequence of easy problems;
  - solve a large problem by solving a sequence of small problems.

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  - Coordinate-search ...

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exactly or approximately, obtaining d_k.

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Step 4. Update the Iterate. If $f(x_k + d_k) < f(x_k)$, then $x_{k+1} := x_k + d_k$; else $x_{k+1} := x_k$. $k := k + 1$. Goto Step 2.

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- $\operatorname{dist}(\nabla f(x_k), \mathcal{S}_k)$ is small enough $\Longleftarrow \mathcal{S}_k \ni \tilde{g}_k \approx \nabla f(x_k)$
- d_k is exact enough $\Longleftarrow$ existing derivative-free algorithms

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4. Numerical results

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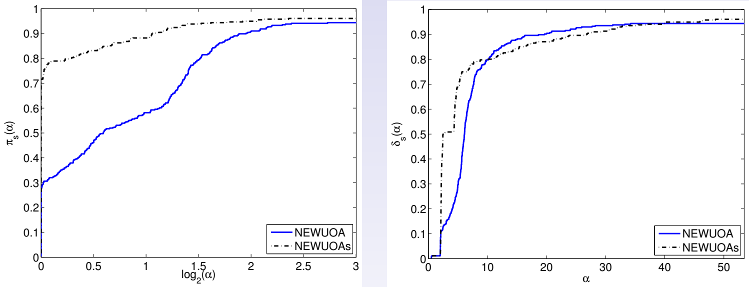
- **Relatively large problems** $(n=50, 100, 150, 200)$



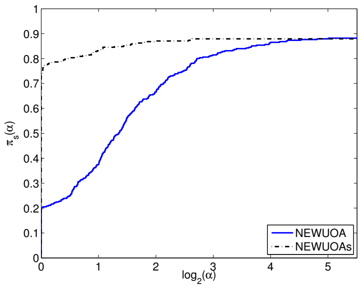
(a) Performance profile

(b) Data profile

Fig 1 : Numerical comparison between `NEWUOA` and `NEWUOAs` $(\tau = 10^{-2})$

4. Numerical results

- **Relatively large problems** ($n = 50, 100, 150, 200$, cont.)



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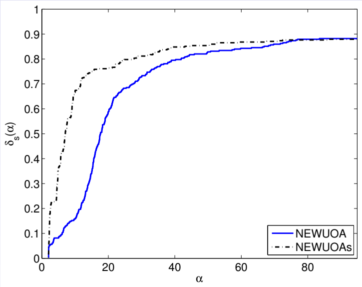

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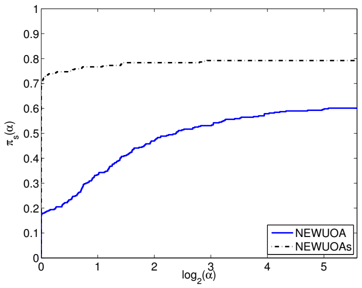
Fig 2 : Numerical comparison between `NEWUOA` and `NEWUOAs` ($\tau = 10^{-4}$)

- **Relatively large problems** ($n = 50, 100, 150, 200$, cont.)



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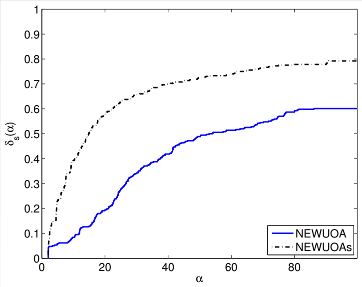

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Fig 3 : Numerical comparison between `NEWUOA` and `NEWUOAs` ($\tau = 10^{-6}$)

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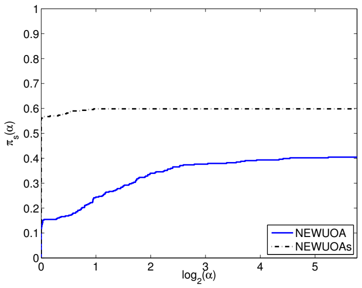

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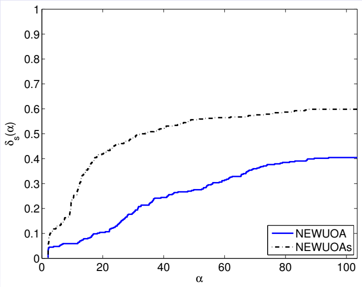

(b) Data profile

Fig 4 : Numerical comparison between `NEWUOA` and `NEWUOAs` ($\tau = 10^{-8}$)

- **Large problems**

Tab 1 : The performance of `NEWUOAs` on some 2000-dimensional problems

	f_{start}	f_{best}	$\#f$	CPU (s)
ARWHEAD	5.997000E+03	0.000000E+00	16095	6.42
BRYBND	7.200000E+04	6.486038E−09	50000	26.09
DIXMAANE	1.471453E+04	1.000000E+00	36264	21.12
DIXMAANF	2.734976E+04	1.000000E+00	36384	31.07
DIXMAANG	5.069653E+04	1.000000E+00	36393	22.72
DQRTIC	6.376035E+15	1.214880E−38	40854	14.70
GENHUMPS	5.122260E+07	1.624799E−26	36467	23.54
LIARWHD	1.170000E+06	2.428807E−24	16208	6.73
POWER	2.668667E+09	1.423292E−11	20130	19.19
SPARSQUR	5.627812E+05	6.381755E−30	16209	9.87

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- General preconditioning techniques

Obrigado!

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Happy birthday, Grandpa!