A Subspace Decomposition Framework for Nonlinear Optimization: Global Convergence and Global Rate

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(Joint work with S. Gratton and Z. Zhang)

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http//www.mat.uc.pt/~lnv

Outline

In this talk, to make things simple:

- we consider unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x);$$

Derivative-free optimization

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Derivative-free optimization

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- we consider unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x);$$

- we suppose that
  - f is smooth, but the derivatives are unavailable.

Derivative-free optimization

- Why derivative-free?

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> *Why work on derivative-free optimization? Because the problems are important and cool.*

— J. Dennis
July 24th, 2013, Toulouse

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  - Directional methods, like direct search

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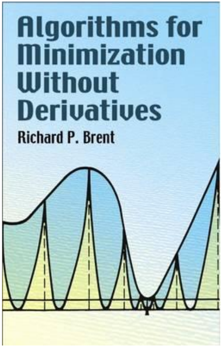
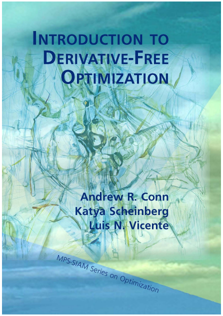
  - Directional methods, like direct search
  - Model-based methods, like trust-region methods

Books





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Difficulty of large-scale problems

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  - difficult to exploit problem structure

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  - divide a difficult problem into a sequence of easy problems, and solve each of them;

  more specifically,
  - divide a large problem into a sequence of small problems, and solve each of them.

An old idea, very old

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分而治之　　　Divide and conquer

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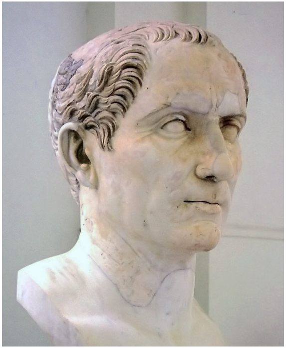

Divide et impera.

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Subspace techniques in optimization

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- Yuan, Ya-xiang. Subspace techniques for nonlinear optimization. Some topics in industrial and applied mathematics 8 (2007): 206-218.

Subspace decomposition techniques in optimization

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- Ferris, Michael C., and Olvi L. Mangasarian. Parallel variable distribution. SIAM Journal on Optimization 4, no. 4 (1994): 815-832.
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- Boyd, Stephen, Lin Xiao, Almir Mutapcic, and Jacob Mattingley. Notes on decomposition methods. Notes for EE364B, Stanford University (2007).
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$$\rho_k = \frac{f(x_k) - f(x_k + d_k)}{\sum_{i=1}^{m_k} \left[f(x_k) - f(x_k + d_k^{(i)}) \right]}$$

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to get $d_k^{(i)}$.

Trust-region framework

Algorithm (Trust-region framework cont.)

Step 4. Obtain d_k by solving

$$\begin{aligned}
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and set Δ_{k+1} so that

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2. *The sequence $\{m_k\}$ is bounded.*
3. *The smallest eigenvalues of $\sum_{i=1}^{m_k} P_k^{(i)}$ are bounded away from zero, where $P_k^{(i)}$ is the orthogonal projection matrix from $\mathbb{R}^n$ onto $\mathcal{S}_k^{(i)}$.*

Global convergence

Theorem

Suppose that the assumptions stated before hold, then the iterates $\{x_k\}$ *generated by either of the frameworks satisfy*

$$\lim_{k \to \infty} \|\nabla f(x_k)\| = 0.$$

Global rate

Theorem

Suppose that the assumptions stated before hold, and additionally

$$\Delta_{k+1} \geq \alpha \Delta_k$$

for some constant $\alpha \in (0, 1]$, *then the iterates* $\{x_k\}$ *generated by the trust-region framework satisfy*

$$\min_{0 \leq \ell \leq k} \|\nabla f(x_\ell)\| \leq C_1 \sqrt{\frac{m}{k}},$$

where m *is an upper bound of* $\{m_k\}$.

Theorem

Suppose that the assumptions stated before hold, and additionally

$$\sigma_{k+1} \leq \beta \sigma_k$$

for some constant $\beta \geq 1$*, then the iterates* $\{x_k\}$ *generated by the Levenberg-Marquardt framework satisfy*

$$\min_{0 \leq \ell \leq k} \|\nabla f(x_\ell)\| \leq C_2 \sqrt{\frac{m}{k}},$$

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Using this and the WCC $O(n^2\varepsilon^{-2})$ for subproblems,

- a reasonable choice for m is $O(\sqrt{n})$
- a reasonable subproblem solution accuracy is $O(n^{-\frac{1}{4}})$

Applications to derivative-free optimization

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$\Downarrow$

Our goal

Parallel and multilevel algorithms without using derivatives and capable of solving relatively large problems.

Very preliminary numerical results

- Use the Levenberg-Marquardt framework
- Subproblem solver: `NEWUOA`
- Number of subspaces: $\sqrt{n/2}$
- Benchmark: `NEWUOA` (`NPT=2N+1; RHOEND=1.0E-6`)
- Very preliminary: not parallel, not multilevel, not large-scale ...
- Dimension of test problems: 25, 30, 35, 40
- Denote our code as `SSD`

VARDIM

Table : Numerical results of VARDIM

n	25	30	35	40	
$\#f$	8343	8926	12689	17741	NEWUOA
	3592	6222	7507	16653	SSD
f_{final}	1.61E-11	4.08E-11	4.93E-11	1.76E-10	NEWUOA
	9.74E-11	6.85E-10	5.74E-11	7.89E-13	SSD

$$f(x) = \sum_{i=1}^{n}(x_i - 1)^2 + \left[\sum_{i=1}^{n} i(x_i - 1)\right]^2 + \left[\sum_{i=1}^{n} i(x_i - 1)\right]^4$$

PENALTY1

Table : Numerical results of PENALTY1

n	25	30	35	40	
$\#f$	9532	10947	14427	13577	NEWUOA
	2089	2784	2348	2812	SSD
f_{final}	2.03E-04	2.48E-04	2.93E-04	3.39E-04	NEWUOA
	2.04E-04	2.50E-04	2.95E-04	3.41E-04	SSD

$$f(x) = 10^{-15} \sum_{i=1}^{n} (x_i - 1)^2 + \left(\frac{1}{4} - \sum_{i=1}^{n} x_i^2 \right)^2$$

SBRYBND

Table : Numerical results of SBRYBND

n	25	30	35	40	
$\#f$	968	576	2052	2363	NEWUOA
	27889	53103	90304	206608	SSD
f_{final}	235	326	342	395	NEWUOA
	3.08	3.08	3.08	3.08	SSD
	134	284	233	229	

$$f(x) = \sum_{i=1}^{n} \left[(2 + 5p_i^2 x_i^2) p_i x_i + 1 - \sum_{j \in J_i} p_j x_j (1 + p_j x_j) \right],$$

where $J_i = \{j \mid j \neq i, \max\{1, i-5\} \leq j \leq \min\{n, j+1\}\}$

CHROSEN

Table : Numerical results of CHROSEN

n	25	30	35	40	
$\#f$	1123	1445	1717	1859	NEWUOA
	96040	103296	127726	142272	SSD
f_{final}	8.94E-12	1.07E-11	1.13E-11	3.14E-11	NEWUOA
	2.95E-10	5.49E-10	7.26E-10	8.09E-10	SSD

$$f(x) = \sum_{i=1}^{n-1} \left[4(x_i - x_{i+1}^2)^2 + (1 - x_{i+1})^2 \right]$$

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- Possible to develop parallel and multilevel methods without using derivatives
- "Clever" way of choosing subspaces ...
  - not try to cover the whole space, but ...
  - choose subspaces randomly

Thanks!

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